Tree-search ML detection for underdetermined MIMO systems with M-PSK constellations

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***Abstract*—In this paper we propose a tree-search algorithm that provides the *exact* ML solution with lower computational complexity than that required by an exhaustive minimum distance search. The new algorithm, that we call *King Decoder*, is based on conditional dominance conditions, a set of sufficient conditions for making optimal decisions regardless of multi-antenna interference. The King Decoder does not require any matrix inversion and/or factorization and can be employed in both underdertermined and overdetermined systems. Complexity performances of the proposed algorithm, obtained through numerical simulations, are compared with those of the generalized sphere decoder, showing a lower search complexity for a wide range of SNR's.**

I. Introduction

Many communication systems can be described by a linear model with additive noise. Examples are narrowband MIMO (multiple-input multiple-output) multiantenna systems [1], multiuser communication systems as in direct-sequence code-division-multiple-access (DS-CDMA) [2] and multi-carrier code-division-multiple-access (MC-CDMA) [3]. The mathematical model is expressed as

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}, \tag{1}$$

where $\mathbf{y} \in \mathbb{C}^N$ is the observation, $\mathbf{H} \in \mathbb{C}^{N \times K}$ is the channel matrix with K inputs and N outputs, $\mathbf{n} \in \mathbb{C}^N$ is a random vector with complex Gaussian distribution with zero mean and variance $2\sigma^2 \mathbf{I}_N$, i.e. $\mathbf{n} \sim \mathcal{CN}\left(0, 2\sigma^2 \mathbf{I}_N\right)$, and $\mathbf{x} \in \mathbb{C}^K$ is the input vector whose elements are drawn from a finite set of complex symbols χ which depends on the specific modulation scheme chosen.

It is well known that the optimal estimation of $\mathbf{x}$ in (1) is the maximum-likelihood (ML) solution

$$\mathbf{x}_{ML} = \arg \min_{\mathbf{x} \in \chi^K} \|\mathbf{y} - \mathbf{H}\mathbf{x}\|_2 . \tag{2}$$

However the search for solution is known to be exponentially complex, as the worst-case computational cost grows exponentially in the number of inputs K and constellation size M (see for example [2]).

A number of suboptimal algorithm have been developed as low-complexity alternatives to the ML decoding. Among others in recent years the sphere decoder [4], [5] has gained considerable attention. Conventional sphere decoding [4] performs well for underloaded systems, i.e. when the size of the input vector K is less then the signal space dimension N, and its use in overloaded or underdetermined systems, i.e. when $K > N$, is not possible. In [5] a Generalized Sphere Decoding (GSD) has been proposed to tackle the problem of optimal decoding of underdetermined systems.

In this paper we first introduce a symbol dominance condition that generalizes to M-PSK the results in [6], [7], [8] for BPSK and QPSK constellations. We then propose a new tree-search algorithm, based on conditional dominance conditions, that gives the ML optimal solution to the problem (2). The algorithm, that we call *King Decoder*, presents lower computational complexity not only with respect to ML exhaustive search, but also to similar tree-search algorithms such the GSD [5]; it does not require any matrix inversion and/or factorization; it can be employed in both underdetermined and overdetermined systems.

The rest of the paper is organized as follows. In Section II we define the discrete difference for the Euclidean distance. In Section III we introduce the symbol dominance condition. In Section IV we present a tree-search algorithm based on conditional dominance conditions and show results of simulations in Section V. Finally Section VI draws the conclusions.

II. Discrete Difference

Geometrically the ML solution is given by the vector $\mathbf{x}$ that minimizes the Euclidean distance

$$f(\mathbf{x}) = (\mathbf{y} - \mathbf{H}\mathbf{x})^H (\mathbf{y} - \mathbf{H}\mathbf{x}) . \tag{3}$$

We are interested in the difference of the Euclidean distance between two generic points of χ^K.

Definition 1: Consider two generic vectors $\mathbf{x}$ and $\hat{\mathbf{x}}$ both belonging to χ^K, we define the *discrete difference* $\Delta f(\mathbf{x}; \hat{\mathbf{x}})$, as the difference $f(\mathbf{x}) - f(\hat{\mathbf{x}})$. The *$k$th discrete difference* along the kth coordinate $\Delta_k f(\mathbf{x}; \hat{\mathbf{x}})$ is the discrete difference $\Delta f(\mathbf{x}; \hat{\mathbf{x}})$ when $\mathbf{x}$ and $\hat{\mathbf{x}}$ differ only by the kth component.

A necessary and sufficient condition for $\mathbf{x}$ to be a global minimum for the cost function $f(\mathbf{x})$ is then that all discrete differences $\Delta f(\mathbf{x}; \hat{\mathbf{x}})$ are non positive for each $\hat{\mathbf{x}} \in \chi^K$. The search of global minimum just by looking at the differences does not reduce the computational complexity of the ML search alone. The number of differences to compute is still exponential with the number of inputs and the size of constellation. However, as it will be clearer in the following, we can avoid to look at all differences and still get the optimal solution.

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In the special case of the Euclidean distance the discrete difference along the generic kth coordinate takes on a specific expression, as stated by the following proposition. We denote with $(\cdot)^H$ the conjugate-transpose operator, with $(\cdot)^*$ the conjugate operator and with $\mathbf{h}_k$ the kth column of matrix $\mathbf{H}$.

Proposition 1: For any pair of vectors $\mathbf{x}$ and $\hat{\mathbf{x}}$ that belong to χ^K and differ only at the kth position

$$\Delta_k f(\mathbf{x};\hat{\mathbf{x}}) = -2\Re\left\{(x_k-\hat{x}_k)^*\left[\mathbf{h}_k^H\mathbf{y}-\sum_{i\neq k}x_i\mathbf{h}_k^H\mathbf{h}_i\right]\right\} + \left(|x_k|^2-|\hat{x}_k|^2\right)\mathbf{h}_k^H\mathbf{h}_k. \quad (4)$$

Proof: Se appendix A. ∎

The expression is quite general and can be used for any system modeled by eq. (1) and for which the detection problem can be formulated as in eq. (2). The kth discrete difference depends on the observed vector $\mathbf{y}$ and on the symbols of the other elements of the input vector $\mathbf{x}$, i.e. x_i, $i\neq k$. The sign of the discrete difference lets us to choose between the two possible transmit vectors $\mathbf{x}$ and $\hat{\mathbf{x}}$. If the discrete difference is non positive then the vector $\mathbf{x}$ is closer to the observation than the vector $\hat{\mathbf{x}}$.

In this paper we restrict our attention on M-PSK constellations, i.e. $\chi=\left\{e^{j\left[\frac{2\pi}{M}(m-1)+\theta\right]}\right\}$, $m=1,\ldots,M$, where θ represents an offset that, without loss of generality, we assume equal to π/M. Since the constellation has constant modulus, we can simplify eq. (4) as

$$\Delta_k f(\mathbf{x};\hat{\mathbf{x}}) = -2\Re\left\{(x_k-\hat{x}_k)^*\left[\mathbf{h}_k^H\mathbf{y}-\sum_{i\neq k}x_i\mathbf{h}_k^H\mathbf{h}_i\right]\right\}. \quad (5)$$

Eq. (5) can be further simplified for all those cases that involve some symmetry between vectors $\mathbf{x}$ and $\hat{\mathbf{x}}$ in the kth dimension. In Fig. 1 a 16-PSK with initial phase offset of $\pi/16$ is shown and each symbol has been denoted with $x_k^{(i)}$ where k is the vector index and i is the index in the constellation set. There exist pairs of symbols that have the same imaginary part, as for example $\left(x_k^{(8)},x_k^{(1)}\right)$, $\left(x_k^{(7)},x_k^{(2)}\right)$, etc., shown in Fig. 1. For all these pairs the discrete difference can be rewritten as

$$\Delta_k f(\mathbf{x};\hat{\mathbf{x}}) = -4\cos\theta_k\Re\left\{\left[\mathbf{h}_k^H\mathbf{y}-\sum_{i\neq k}x_i\mathbf{h}_k^H\mathbf{h}_i\right]\right\}, \quad (6)$$

where θ_k is the phase corresponding to the symbol x_k. Since we are interested in the sign of the discrete difference, the above expression allows to exclude all points for which the discrete difference is positive. For example, if for $x_k^{(1)}$ $\Delta_k f(\mathbf{x};\hat{\mathbf{x}})>0$, given observation $\mathbf{y}$ and the other elements of the input vector x_i, then we can conclude that the symbol $x_k^{(1)}$ cannot be a valid solution for the kth component of the ML solution. In other words, under the assumption that we already know x_i, $i\neq k$ and given the observation $\mathbf{y}$, just

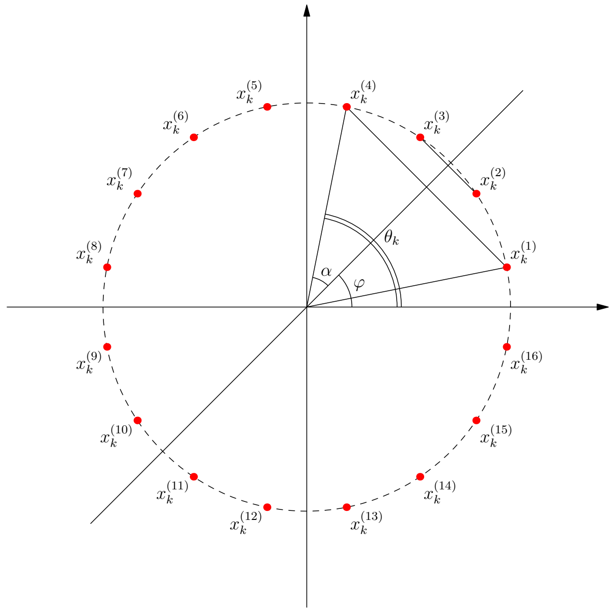

Figure 1. 16-PSK constellation for the kth component of the input vector $\mathbf{x}$. The line at angle φ splits the constellation set into two subsets. By looking at the sign of the discrete difference for pair of symbols that are symmetric with respect to the line we can select the subset which the kth component of the optimal ML solution belongs to.

by looking at the sign of the discrete difference (6) we can exclude half the symbols that belong to the constellations. To see this note that

$$\text{sign}(\cos\theta_k) = \text{sign}\left\{\Re\left\{\left[\mathbf{h}_k^H\mathbf{y}-\sum_{i\neq k}x_i\mathbf{h}_k^H\mathbf{h}_i\right]\right\}\right\}. \quad (7)$$

is a necessary and sufficient condition for the sign of the discrete difference between vectors that differ in the kth coordinate and have the same imaginary part. Therefore the sign of the real part of the kth component of the optimal solution is completely determined by the sign of the real part of $\left[\mathbf{h}_k^H\mathbf{y}-\sum_{i\neq k}x_i\mathbf{h}_k^H\mathbf{h}_i\right]$. An analogous condition for pairs of symbols that have the same real part is given by

$$\text{sign}(\sin\theta_k) = \text{sign}\left\{\Re\left\{e^{-j\frac{\pi}{2}}\left[\mathbf{h}_k^H\mathbf{y}-\sum_{i\neq k}x_i\mathbf{h}_k^H\mathbf{h}_i\right]\right\}\right\}. \quad (8)$$

Therefore the sign of the imaginary part of the kth component of the optimal solution is fully determined by the sign of the real part of the complex point rotated by $-\pi/2$.

The most general case is shown in Fig. 1 where a 16-PSK is shown and a line that symmetrically splits the constellation set into two subsets is also shown. The angle between the line and the real axis is denoted by φ. The discrete difference for pairs of symbols that are symmetric with respect to the line at angle φ is given in the following proposition.

Proposition 2: Consider a line in the complex plane that forms an angle φ with the real axis that splits symmetrically a

M-PSK constellation into two subsets. For any pair of vectors $\mathbf{x}$ and $\hat{\mathbf{x}}$, both belonging to χ^K, that differ only by the kth component and symmetric with respect to the line at angle φ the discrete difference is given by

$$\Delta_k f(\mathbf{x};\hat{\mathbf{x}}) = \\ -4\sin(\theta_k-\varphi)\,\Re\left\{e^{-j\left(\varphi+\frac{\pi}{2}\right)}\left[\mathbf{h}_k^H\mathbf{y}-\sum_{i\neq k}x_i\mathbf{h}_k^H\mathbf{h}_i\right]\right\}, \tag{9}$$

where θ_k is the phase of the kth component of the vector $\mathbf{x}$.

Proof: See appendix B. ■

The general expression (9) contains also the specific expression given, for example, by eq. (6), by simply choosing $\varphi = 0$. The analogous equation for pairs with same real part can be obtained by choosing $\varphi = \pi/2$.

A necessary and sufficient condition for the sign of the discrete difference can also be derived from eq. (9). The sign of the discrete difference is non positive iff

$$\operatorname{sign}(\sin(\theta_k-\varphi)) = \\ \operatorname{sign}\left\{\Re\left\{e^{-j\left(\varphi+\frac{\pi}{2}\right)}\left[\mathbf{h}_k^H\mathbf{y}-\sum_{i\neq k}x_i\mathbf{h}_k^H\mathbf{h}_i\right]\right\}\right\}. \tag{10}$$

The above equation generalizes conditions (7) and (8) and has the following geometrical interpretation. Given an observation $\mathbf{y}$ and all other input vector components x_i, $i \neq k$, the above equation allows to determine if the phase of kth component of the optimal solution is in the interval $(\varphi, \varphi+\pi)$. Note that this is equivalent to exclude one of the two subsets in which the line at angle φ splits the constellation. For example, in Fig. (1), we have that if the sign of the term on the right-hand side of eq. (10) is positive then the kth component of the optimal solution is in the set of symbols $\left\{x_k^{(3)}, x_k^{(4)}, \ldots, x_k^{(9)}, x_k^{(10)}\right\}$.

A single condition of the type given by eq. (10) is not in general sufficient for making an optimal decision on the kth component of the transmitted vector $\mathbf{x}$. However it can be noted that a combination of equations of type (10) with different angles φ is sufficient to make an optimal decision, if all x_i, $i \neq k$, are known. Therefore by using eq. (10) the optimal decision on the kth component of the vector $\mathbf{x}$ can be equivalently made by a set of properly defined binary decisions, just by looking at signs.

III. Dominance condition

Eq. (9) can be used to make an optimal decision under the assumption that the contribution due to the other components of vector $\mathbf{x}$ are known. Unfortunately this is not always true as the detection problem aims just at the estimation of this contribution. However we can still identify cases where the determination of the sign of the kth discrete difference can be made regardless of the contribution of all other components of $\mathbf{x}$. A sufficient condition for the determination of the sign of the kth discrete difference is given by the following proposition.

Proposition 3: If the following condition is satisfied

$$\left|\Re\left\{\mathbf{h}_k^H\mathbf{y}e^{-j\left(\varphi+\frac{\pi}{2}\right)}\right\}\right| > \sum_{i\neq k}\left|\mathbf{h}_k^H\mathbf{h}_i\right| \tag{11}$$

then the sign of the corresponding kth discrete difference is determined regardless of the contribution of all other components of $\mathbf{x}$.

Proof: See appendix C. ■

Inequality (11) is a *dominance condition* because, when it holds the kth component of the projected received vector is so strong that dominates all other components. Eq. (11) is a generalization to M-PSK of the bit dominance condition that was first introduced in [6] and then in [7], [8].

The dominance condition assumes that in eq. (11) no symbols x_i, $i \neq k$, are known. However, during decoding, partial knowledge may be available. In such cases the sign of the discrete difference depends only on the subset of x_i that are still to be decoded. A dominance condition when only a subset $\mathcal{W}$ of symbols is already available, can be given.

Proposition 4: Given the set of known symbols $\mathcal{W}$ and a set of unknown symbols $\mathcal{O}$, if the following condition holds

$$\left|\Re\left\{e^{-j\left(\varphi+\frac{\pi}{2}\right)}\left[\mathbf{h}_k^H\mathbf{y}-\sum_{m\in\mathcal{W},m\neq k}x_m\mathbf{h}_k^H\mathbf{h}_m\right]\right\}\right| > \\ \sum_{i\in\mathcal{O},i\neq k}\left|\mathbf{h}_i^H\mathbf{h}_k\right|, \tag{12}$$

then the sign of the corresponding kth discrete difference is determined regardless of the contribution of all components of $\mathbf{x}$, x_i, $i \in \mathcal{O}$.

Proof: Analogous to the proof of Prop. 3.. ■

IV. Tree-search algorithm

We consider the decoding process as a decision on a tree with $K+1$ layers, $|\chi|$ branches that departs from each non leaf node and $|\chi|^K$ leaf nodes corresponding to all possible transmit vectors $\mathbf{x}$. An exhaustive search of the optimal solution would require to traverse the entire tree. However if we could make decisions on possibly each node and, as consequence, cut some node, the number of visited nodes would be reduced.

We then associate to each node a set of conditional dominance conditions given the symbols corresponding to parent nodes. For example, for a 8-PSK constellation the set of conditions (12) for $\varphi = 0$, $\varphi = \pi/2$, $\varphi = \pi/4$ and $\varphi = -\pi/4$ can be considered at each node. Therefore a tree-search algorithm can be formulated based on the check at each node of the corresponding set of conditional dominance conditions. If any conditional dominance condition is satisfied then half constellation symbols can be excluded by the set of possible symbols for the optimal solution. If all conditions are satisfied then an optimal decision on the corresponding symbol can be made and $|\chi|-1$ branches can be cut. At the end of the search process we obtain only a subset of χ^K , among which we have the optimal solution.

To select the optimal solution the Euclidean distance must be computed. In order to reduce the computational complexity required for this last step we introduce a different equivalent metric $d(x_1, \dots, x_K)$, that has two attracting properties: part of the metric is already computed when the dominance conditions are computed; the metric is *cumulative,* i.e. if two candidates on the tree have part of the path in common, then they have a common "partial metric", so enabling the computation of the metrics through partial updates on the nodes of the tree.

We rewrite eq. (2) as

$$\mathbf{x}_{ML} = \arg\max_{\mathbf{x}\in\chi^K} 2\Re\left\{\mathbf{x}^H\mathbf{H}^H\mathbf{y}\right\} - \mathbf{x}^H\mathbf{H}^H\mathbf{H}\mathbf{x}. \quad (13)$$

Under the assumption of constant modulus constellation, as for M-PSK, the above problem does not change if we add the term $\mathbf{x}^H\mathbf{E}\mathbf{x}$, where $\mathbf{E}$ is a diagonal matrix whose diagonal elements are $e_k = \mathbf{h}_k^H\mathbf{h}_k$, as $\mathbf{x}^H\mathbf{E}\mathbf{x} = \sum_{k=1}^K \|\mathbf{h}_k\|_2$. Therefore we obtain

$$\mathbf{x}_{ML} = \arg\max_{\mathbf{x}\in\chi^K} 2\Re\left\{\mathbf{x}^H\mathbf{H}^H\mathbf{y}\right\} - \mathbf{x}^H\mathbf{H}^H\mathbf{H}\mathbf{x} + \mathbf{x}^H\mathbf{E}\mathbf{x}. \quad (14)$$

Since $\mathbf{H}^H\mathbf{H} - \mathbf{E}$ is Hermitian, we can write it as the sum $\mathbf{L} + \mathbf{L}^H$, where $\mathbf{L}$ is the lower triangular part of $\mathbf{H}^H\mathbf{H} - \mathbf{E}$, and obtain

$$\begin{aligned}\mathbf{x}_{ML} &= \arg\max_{\mathbf{x}\in\chi^K} \Re\left\{2\mathbf{x}^H\mathbf{H}^H\mathbf{y} - \mathbf{x}^H\left(\mathbf{L} + \mathbf{L}^H\right)\mathbf{x}\right\} && (15)\\ &= \arg\max_{\mathbf{x}\in\chi^K} \Re\left\{\mathbf{x}^H\left[\mathbf{H}^H\mathbf{y} - \mathbf{L}\mathbf{x}\right]\right\}. && (16)\end{aligned}$$

The metric obtained can be expressed as

$$d(x_1, \dots, x_K) = \Re\left\{\sum_{k=1}^K x_k^* \left[\mathbf{h}_k^H\mathbf{y} - \sum_{i=1}^{k-1} l_{ki}x_i\right]\right\}, \quad (17)$$

where the term in parenthesis is already available from the conditional dominance condition. The metric is also cumulative because the inner summation is done only on already visited nodes.

The use of the cumulative metric that we have introduced can save computational power and allows to terminate the tree-search as soon as no more nodes need to be visited, because the optimal solution is just the leaf node with the best metric.

The algorithm that we propose is a generalization, and computationally more efficient version, of the King Decoder algorithm that has been proposed in [8], whose formulation was restricted to real-valued models and binary constellations. The name is King Decoder, because it is based on the generalized symbol conditional *dominance* condition (12).

V. Simulations

Since the King Decoder provides the optimal solution, we evaluate its performances in terms of average search complexity, measured as the number of visited nodes, as suggested by [9], by means of Monte-Carlo simulations. We consider an underdetermined flat-fading multiantenna MIMO system with $N = 2$ receive and $K = 4$ transmit antennas, 8-PSK modulation with an initial offset of $\pi/8$ and a random channel matrix with each element $h_{ij} \sim \mathcal{CN}(0,1)$ representing the fading between transmitter j and receiver i.

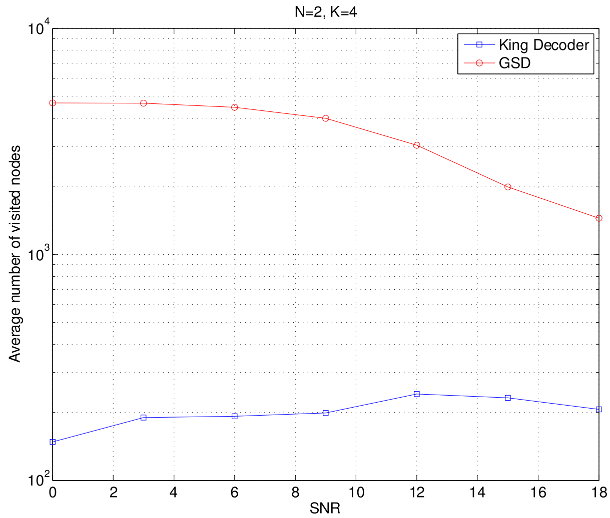

Figure 2. Comparison of the average number of visited nodes for King Decoder and Generalized Sphere Decoder for a MIMO system with $N = 2$ receive and $K = 4$ transmit antennas and 8-PSK constellation. SNR is the total average transmitted power over a symbol period over the additive Gaussian noise power [1].

We also compare our algorithm with GSD [5], as it represents an efficient version of the Sphere Decoder [4] that can be applied to rank deficient systems. We have implemented the GSD with a Fincke-Pohst strategy [10] and a starting radius $r^2 = 7N\sigma^2$, with a restart with increased radius if no vectors are found within the sphere and therefore it provides the optimal solution.

In Fig. 2 results, in terms of average number of visited nodes versus signal-to noise ratio (SNR), show that the King Decoder has better performances than GSD for the entire range of SNR's considered. It is also interesting to note that King decoder presents an almost constant complexity. The reason is that the impact of the channel's structure is more pronounced than that of the channel noise.

VI. Conclusions

In this paper we have introduced symbol dominance conditions that represent sufficient conditions for making optimal decisions in MIMO systems with M-PSK constellations. We have also shown that conditional dominance conditions can be used in a tree-search algorithm, that we have called King Decoder, capable of ML decoding at reduced complexity. The King Decoder has several advantages: the same algorithm can be employed in both underdetermined and overdetermined systems, in contrast to conventional sphere decoding; no preprocessing is required as in sphere decoding, in particular no matrix inversion and/or factorization is needed; simulation results show that it presents lower complexity than the Generalized Sphere Decoder.

APPENDIX A
PROOF OF PROPOSITION 1

We explicitly write the difference as:

$$\Delta_k f(\mathbf{x};\hat{\mathbf{x}}) = -2\Re\left\{(x_k-\hat{x}_k)^* \mathbf{h}_k^H \mathbf{y}\right\} + \mathbf{x}^H\mathbf{H}^H\mathbf{H}\mathbf{x} - \hat{\mathbf{x}}^H\mathbf{H}^H\mathbf{H}\hat{\mathbf{x}} \quad (18)$$

The term $\mathbf{x}^H\mathbf{H}^H\mathbf{H}\mathbf{x} - \hat{\mathbf{x}}^H\mathbf{H}^H\mathbf{H}\hat{\mathbf{x}}$ is a real scalar, so we can apply the conjugate-transpose operator with no change to obtain

$$\begin{aligned}&\mathbf{x}^H\mathbf{H}^H\mathbf{H}\mathbf{x} - \hat{\mathbf{x}}^H\mathbf{H}^H\mathbf{H}\hat{\mathbf{x}} = \\ &\Re\left\{\sum_i\sum_j x_i^*\mathbf{h}_i^H\mathbf{h}_j x_j - \sum_m\sum_n \hat{x}_m^*\mathbf{h}_m^H\mathbf{h}_n\hat{x}_n\right\} = \\ &2\Re\left\{(x_k-\hat{x}_k)^*\left[\mathbf{h}_k^H\mathbf{y} - \sum_{i\neq k} x_i\mathbf{h}_k^H\mathbf{h}_i\right]\right\} \\ &+ \left(|x_k|^2 - |\hat{x}_k|^2\right)\mathbf{h}_k^H\mathbf{h}_k \quad (19)\end{aligned}$$

We can finally write the difference as stated by the proposition.

APPENDIX B
PROOF OF PROPOSITION 2

Consider the discrete difference given by eq. (5). When $\hat{x}_k$ is symmetric with respect to the line at angle φ, the difference $x_k - \hat{x}_k$ in eq. (5) can be rewritten in terms of θ_k and φ. In Fig. (1) pairs $\left(x_k^{(4)}, x_k^{(1)}\right)$ and $\left(x_k^{(3)}, x_k^{(2)}\right)$ are examples of the kind of differences that we are considering. For all these pairs the difference becomes

$$\begin{aligned}(x_k-\hat{x}_k)^* &= \left(e^{j(\varphi+\alpha)} - e^{j(\varphi-\alpha)}\right)^* &(20)\\ &= 2e^{-j\left(\varphi+\frac{\pi}{2}\right)}\sin\alpha &(21)\end{aligned}$$

Since we have $\alpha = \theta_k - \varphi$, we finally have the eq. (11).

APPENDIX C
PROOF OF PROPOSITION 3

The sign of the discrete difference is determined regardless of the contribution of all other components of $\mathbf{x}$ whenever the following condition holds

$$\left|\Re\left\{\mathbf{h}_k^H\mathbf{y}e^{-j\left(\varphi+\frac{\pi}{2}\right)}\right\}\right| > \left|\Re\left\{\sum_{i\neq k} x_i\mathbf{h}_k^H\mathbf{h}_i e^{-j\left(\varphi+\frac{\pi}{2}\right)}\right\}\right|. \quad (22)$$

For the triangle inequality a stronger condition is expressed by the following inequality

$$\left|\Re\left\{\mathbf{h}_k^H\mathbf{y}e^{-j\left(\varphi+\frac{\pi}{2}\right)}\right\}\right| > \sum_{i\neq k}\left|\Re\left\{x_i\mathbf{h}_k^H\mathbf{h}_i e^{-j\left(\varphi+\frac{\pi}{2}\right)}\right\}\right|. \quad (23)$$

An even stronger inequality is obtained by observing that the real part of a complex number is maximum when equals its modulus

$$\left|\Re\left\{\mathbf{h}_k^H\mathbf{y}e^{-j\left(\varphi+\frac{\pi}{2}\right)}\right\}\right| > \sum_{i\neq k}\left|x_i\mathbf{h}_k^H\mathbf{h}_i e^{-j\left(\varphi+\frac{\pi}{2}\right)}\right|. \quad (24)$$

Under the assumption of constant modulus constellation the above inequality can be rewritten as

$$\left|\Re\left\{\mathbf{h}_k^H\mathbf{y}e^{-j\left(\varphi+\frac{\pi}{2}\right)}\right\}\right| > \sum_{i\neq k}\left|\mathbf{h}_k^H\mathbf{h}_i\right|. \quad (25)$$

By writing explicitly the real part of the left-hand term we finally get the sufficient condition stated by the proposition.

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